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THE RELATIVE PLURICANONICAL STABILITY FOR 3-FOLDS OF GENERAL TYPE, II

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Abstract

The aim of this paper is to study the pluricanonical maps of smooth projective 3-folds of general type. For a given 3-fold X of general type, define k_0 to be the minimal integer such that the k_0-th plurigenus $P_{k_0}(X) := h^0(X, k_0K_X) \geq 2$, Kollár proved that the $(11k_0+5)$-canonical map is birational. However, given an arbitrary integer $m > 11k_0+5$, it is hard to know from Kollár's method whether the m-canonical map is still birational or not. On the basis of our previous works, we shall prove, by developing a new approach, that the $(7k_0+3)$-canonical map is birational and that the m-canonical map is birational whenever $m \geq 10k_0+6$. If $k_0 \geq 25$, then we shall show that the m-canonical map is birational whenever $m \geq 8k_0+6$. Furthermore, if X is irregular (i.e. $h^1(\mathcal{O}_X) > 0$), then the m-canonical map is birational whenever $m \geq 166$.

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To classify algebraic varieties is one of the goals of algebraic geometry. Let X be a smooth projective variety of dimension d, K_X be the canonical divisor and ω_X the dualizing sheaf. When the system $|mK_X| \neq \emptyset$, we can define a natural rational map

$$\phi_m := \Phi_{|mK_X|} : X \dashrightarrow \mathbb{P}^{P_m(X)-1}$$

where $P_m(X) := h^0(X, \omega_X^{\otimes m})$ is called *the m-th plurigenus* of X and ϕ_m is called *the m-th pluricanonical map.* It is obvious that the behavior of ϕ_m directly reflects intrinsic properties of X, so that studying the pluricanonical maps is quite important to the classification theory. Usually, people are curious about whether ϕ_m is an embedding, a birational map, a generically finite map or a map of fiber type. Furthermore, if it is generically finite, what is the variety downstairs and what is the degree of the cover? If it is of fiber type, what is the base variety and what is a general fiber? These questions help to understand the behavior of ϕ_m. The objects considered in this paper are supposed to be varieties of general type. When $d = 1$, a smooth projective curve X of general type has the genus $g(X) \geq 2$. The behavior of its pluricanonical maps is quite clear. Explicitly, ϕ_m is always an embedding whenever $m \geq 3$. ϕ_2 is an embedding with the only exception of genus two case when it is a double cover. According to the behavior of ϕ_1, X is called *a hyper-elliptic curve* if ϕ_1 is a double cover, *a non-hyper-elliptic curve* if ϕ_1 is an embedding. When $d = 2$, the situation is more complicated, however, the behaviors of ϕ_m are almost clear by virtue of a great deal of works by many authors. Since this is not a survey article, we don't plan to mention more references here. Instead, the results which will be applied in our argument can be found in [Bo], [B-C], [Ca], [Ci], [Mi], [Rr], and [X1], etc. It is wellknown that ϕ_m is birational whenever $m \geq 5$, that ϕ_4 is birational with the exception for surfaces with $(K^2, p_g) = (1, 2)$, that ϕ_3 is birational with the exception for surfaces with $(K^2, p_g) = (1, 2)$ or $(2, 3)$, and that ϕ_2 is generically finite with the exception for surfaces with $(K^2, p_g) = (1, 0)$.

It is natural that one should ask about the status of study in the case of $d \geq 3$. As far as we know, it remains open whether there is a constant $m_0(d)$ such that ϕ_m is birational for any smooth projective d-fold of general type whenever $m \geq m_0(d)$. Comparing with the surface case, we lack of an effective plurigenera, although the 3-dimensional minimal model theory has already been well established. To fix the terminology, we say that ϕ_{m_0} is *stably birational* if ϕ_m is birational whenever $m \geq m_0$. A very natural question (Question 3.2 of [Ch]) arises:

does "birational" imply "stably birational"?

This is quite non-trivial, though it is true in the case of $d \leq 2$. Since X is supposed to be of general type, ϕ_m is stably birational whenever $m \gg 0$. So the first step is to find an optimal bound for this m, once given a variety X. We need the following definition.

Definition 0.1. Let X be a nonsingular projective variety of general type of dimension d. We define

$k_0(X) := min\{k|\ k \in \mathbb{Z}^+,\ P_k(X) \geq 2\}$;

$k_s(X) := min\{k|\ k \in \mathbb{Z}^+,\ \phi_k$ is stably birational$\}$;

$\mu_s(X) := \frac{k_s(X)}{k_0(X)}$, which is called *the relative pluricanonical stability* of X. Obviously, $\mu_s(X)$ is a birational invariant.

$\mu_s(d) := \sup\{\mu_s(X) |\ X$ runs through all smooth projective d-folds of general type$\}$, which is called the d-th *relative pluricanonical stability.*

Noting that $k_0(X)$ is intrinsic with respect to the given X and $k_0(X) < +\infty$, it is reasonable to study ϕ_m in the relative way, i.e. to find the optimal bound for $k_s(X)$ in terms of $k_0(X)$. The invariant $k_s(X)$ is important because it is not only crucial to the classification theory, but also strongly related to other interesting problems. For example, it can be applied to determine the order of the birational automorphism group of X ([X2], Remark in §1). According to [Ko] and [Ch], one has the following

Known Results. *Let X be a smooth projective 3-fold of general type, denote $k_0 := k_0(X)$, then*

(R1) ([Ko, Corollary 4.8]*) ϕ_{11k_0+5} is birational;*

(R2) ([Ch, Main Theorem]*) either ϕ_{7k_0+3} or ϕ_{7k_0+5} is birational and ϕ_{13k_0+6} is stably birational, so $\mu_s(3) \leq 16$;*

(R3) ([Ch, Corollary 2.3.1], [F, Theorem 4.2], [Ko, Remark 6.6]*) if X is irregular (i.e. $h^1(\mathcal{O}_X) > 0$), then ϕ_{143} is birational.*

With a new idea, we aim to present much better bounds here which greatly improve known results. We shall study, case by case, the following questions.

Q1. If ϕ_{k_0} is birational, when is $\phi_{m(k_0)}$ stably birational, where $m(k_0)$ is a function in terms of k_0?

Q2. If $\dim\phi_{k_0}(X) = 3$ and ϕ_{k_0} is not birational, when is $\phi_{m_3(k_0)}$ stably birational, where $m_3(k_0)$ is a function in terms of k_0?

Q3. If $\dim\phi_{k_0}(X) = n$, $1 \leq n \leq 2$, when is $\phi_{m_n(k_0)}$ stably birational, where $m_n(k_0)$ is a function in terms of k_0 for each n?

The main consequences of our technique are the following

Main Results. *Let X be a smooth projective 3-fold of general type, denote $k_0 := k_0(X)$, then*

(i) ϕ_{7k_0+3} is birational.

(ii) ϕ_{10k_0+6} is stably birational and thus $\mu_s(3) \leq 13$; if $k_0 \geq 25$, then ϕ_{8k_0+6} is stably birational.

(iii) if $q := h^1(\mathcal{O}_X) > 0$, then ϕ_{166} is stably birational; if either $q > 1$ or $q = 1$ but $\chi(\mathcal{O}_X) \neq 1$, then ϕ_{125} is stably birational.

These results are contained in Theorem 3.3, Theorem 3.4, Theorem 3.5, Theorem 3.7, Theorem 3.9, Theorem 3.10, Corollary 4.4, Corollary 4.5 and Corollary 4.6.

The reason of my writing this paper is that the whole setting and the main approach here are quite different from those in my previous one. On the other hand, we feel that the above results are closer to the optimal ones which some experts ever expected. It is very strange to me that the stable bound k_s obtained in this paper is even better than Kollár's birational bound $11k_0 + 5$. For the reader's convenience, we try to arrange the whole argument to be self-contained. The method of this paper is a development to the traditional one. First we use the Kawamata-Viehweg vanishing theorem to reduce the problem to a parallel one for the adjoint system $|K_S + L|$ on a smooth projective surface S of general type. In general, I. Reider's result cannot be applied to this system since L is not a nef and big Cartier divisor, instead L is the round-up of a nef and big $\mathbb{Q}$-divisor A, i.e. $L = \ulcorner A \urcorner$. We are not going to treat a very general case since it is difficult to do so. Thanks to expected properties of the divisor A, we managed to find a sufficient condition for the birationality of the system $|K_S+L|$. However, the difficult step is to find a suitable A or L which satisfies this condition.

1. Preliminaries

Throughout this paper, the ground field is supposed to be any algebraically closed field of characteristic zero. Let X be a normal projective variety of dimension d. We denote by $\mathrm{Div}(X)$ the group of Weil divisors on X. An element $D \in \mathrm{Div}(X) \otimes \mathbb{Q}$ is called a *$\mathbb{Q}$-divisor*. A $\mathbb{Q}$-divisor D is said to be *$\mathbb{Q}$-Cartier* if mD is a Cartier divisor for some positive integer m. For a $\mathbb{Q}$-Cartier divisor D and an irreducible curve $C \subset X$, we can define the intersection number $D \cdot C$ in a natural way. A $\mathbb{Q}$-Cartier divisor D is called *nef* (or *numerically effective*) if $D \cdot C \geq 0$ for any effective curve $C \subset X$. A nef divisor D is called *big* if $D^d > 0$. We say that X is *$\mathbb{Q}$-factorial* if every Weil divisor on X is $\mathbb{Q}$-Cartier. For a Weil divisor D on X, write $\mathcal{O}_X(D)$ as the corresponding reflexive sheaf. Denote by K_X a canonical divisor of X, which is a Weil divisor. X is called *minimal* if K_X is a nef $\mathbb{Q}$-Cartier divisor. For a positive integer m, we set $\omega^{[m]} := \mathcal{O}_X(mK_X)$ and call $P_m(X) := \dim_{\mathbb{C}} H^0(X, \omega^{[m]})$ the *m-th plurigenus* of X. We remark that $P_m(X)$ is an important birational invariant. Define the Kodaira dimension $\mathrm{kod}(X)$ to be k, $1 \leq k \leq \dim X$, if there are two constants α and β such that

$$\alpha m^k < P_m(X) < \beta m^k, \quad \text{for } m \gg 0.$$

X is said to be of general type if $\mathrm{kod}(X) = \dim X$.

X is said to have only *canonical singularities* (resp. *terminal singularities*) according to Reid ([R]) if the following two conditions hold:

(i) for some positive integer r, rK_X is Cartier;

(ii) for some resolution $f: Y \longrightarrow X$, $K_Y = f^*(K_X) + \sum a_i E_i$ for $0 \leq a_i \in \mathbb{Q}$ (resp. $0 < a_i$) $\forall i$, where the E_i vary all the exceptional divisors on Y.

According to 3-dimensional MMP ([KMM], [K-M]), when V is a smooth projective threefold of positive Kodaira dimension, there exists a birational map $\sigma : V \dashrightarrow X$, where X can be a minimal 3-fold with only $\mathbb{Q}$-factorial terminal singularities and σ is a composite of successive divisorial contractions and flips. Usually, X is not uniquely determined by V.

Let $D = \sum a_i D_i$ be a $\mathbb{Q}$-divisor on X where the D_i are distinct prime divisors and $a_i \in \mathbb{Q}$. We define

the round-down $\llcorner D \lrcorner := \sum \llcorner a_i \lrcorner D_i$, where $\llcorner a_i \lrcorner$ is the integral part of a_i,

the round-up $\ulcorner D \urcorner := -\llcorner -D \lrcorner$,

the fractional part $\{D\} := \ulcorner (D - \llcorner D \lrcorner) \urcorner$.

Remark 1.1. Suppose X has only canonical singularities and $f: V \longrightarrow X$ is a resolution, we have

$$P_m(X) = h^0\Big(V, \mathcal{O}_V\big(\llcorner f^*(mK_X) \lrcorner\big)\Big) = h^0\Big(V, \mathcal{O}_V\big(\ulcorner f^*(mK_X) \urcorner\big)\Big) = P_m(V)$$

for any positive integer m.

Though it seems that the next definition is not standard, we would rather give it in order to avoid unnecessary redundancy throughout the whole context.

Definition 1.2. Let X be a smooth projective variety and L be a Cartier divisor on X. If $|L|$ is a linear system without fixed components and $h^0(X, L) \geq 2$, we mean *a generic irreducible element* S of $|L|$ as follows:

(i) if $\dim \Phi_{|L|}(X) \geq 2$, then S is just a general member of $|L|$.

(ii) if $\dim \Phi_{|L|}(X) = 1$, then L is linearly equivalent to a union of distinct reduced irreducible divisors of the same type. Explicitly, $L \sim_{\text{lin}} \sum S_i$. We mean S a generic S_i.

We always use the Kawamata-Ramanujam-Viehweg vanishing theorem in the following form.

1.3 Vanishing Theorem. *(*[Ka] *or* [V]*) Let X be a smooth complete variety, D is a $\mathbb{Q}$-divisor. Assume the following two conditions:*

(i) D is nef and big;

(ii) the fractional part of D has supports with only normal crossings.

Then $H^i(X, \mathcal{O}_X(K_X + \ulcorner D \urcorner)) = 0$ for all $i > 0$.

(We remark that the normal crossing property is unnecessary when X is an algebraic surface, by virtue of Sakai's result.)

1.4 The Matsuki-Tankeev principle. This principle is tacitly used throughout our argument. Suppose X is a smooth variety, $|M|$ is a base point free system on X and D is a divisor with $|D| \neq \varnothing$. We want to know when $\Phi_{|D+M|}$ is birational. The following principles are due to Tankeev and Matsuki, respectively.

(P1). *(Lemma 2 of* [T]*) Suppose $|M|$ is not composed of a pencil, i.e.* $\dim \Phi_{|M|}(X) \geq 2$ *and take a general member $Y \in |M|$. If the restriction of $\Phi_{|D+M|}$ to Y is birational, then $\Phi_{|D+M|}$ is birational.*

(P2). *(see* [Ma]*) Suppose $|M|$ is composed of a pencil and take the Stein-factorization of*

$$\Phi_{|M|} : X \xrightarrow{f} C \longrightarrow W \subset \mathbb{P}^N,$$

where W is the image of X through $\Phi_{|M|}$ and f is a fibration onto a smooth curve C. Let F be a general fiber of f. If we have known (say by the vanishing theorem) that $\Phi_{|D+M|}$ can distinguish general fibers of f and that its restriction to F is birational, then $\Phi_{|D+M|}$ is also birational.

1.5 Kollár's technique. This approach comes from [Ko]. In some cases, its output is better and is, sometimes, applied to our arguments. Let X be a smooth projective 3-fold of general type and suppose $P_k(X) \geq 2$. Choose a 1-dimensional sub-system of $|kK_X|$ and replace X by a birational model X' where this pencil defines a morphism $g : X' \longrightarrow \mathbb{P}^1$. (For simplicity, we can suppose $X' = X$). Let S be a generic irreducible element of this pencil, then a general fiber of g is a disjoint union of some surfaces with the same type as S and S is a smooth projective surface of general type. Let $t = k(2p+1) + p$. Then $H^0(\omega_X^t) = H^0(\mathbb{P}^1, g_*\omega_X^t)$ and we have an injection $\mathcal{O}(1) \hookrightarrow g_*\omega_X^k$, and hence an injection $\mathcal{O}(2p+1) \hookrightarrow g_*\omega_X^{k(2p+1)}$. This gives an injection

$$\mathcal{O}(2p+1) \otimes g_*\omega_X^p \hookrightarrow g_*\omega_X^t,$$

where $\mathcal{O}(2p+1)\otimes g_*\omega_X^p = \mathcal{O}(1)\otimes g_*\omega_{X/\mathbb{P}^1}^p$. Now it is wellknown that $g_*\omega_{X/\mathbb{P}^1}^p$ is a sum of line bundles of non-negative degree on $\mathbb{P}^1$. If $p \geq 5$, the local sections of $g_*\omega_X^p$ give a birational map for S, and all these extend to global sections of $\mathcal{O}(2p+1)\otimes g_*\omega_X^p$. Moreover its sections separate the fibers from each other, hence ϕ_t is a birational map for X whenever $p \geq 5$. From this method, according to [BPV] and [X], we can see

(1.5.1) ϕ_{5k+2} is generically finite for X if S is not a surface with $p_g(S) = q(S) = 0$ and $K_{S_0}^2 = 1$, where S_0 is the minimal model of S. Otherwise, we have at least $\dim\phi_{5k+2}(X) \geq 2$.

(1.5.2) ϕ_{7k+3} is birational for X if S is not a surface with

$$(K_{S_0}^2, p_g(S)) = (1,2) \text{ or } (2,3).$$

2. Several Lemmas

Lemma 2.1. *Let S be a smooth projective surface of general type, L be a nef and big Cartier divisor on S, then*

(i) $\Phi_{|K_S+mL|}$ is birational if $m \geq 4$;

(ii) $\Phi_{|K_S+3L|}$ is birational if $L^2 \geq 2$;

(iii) $K_S + D$ is effective if D is a divisor with $h^0(S,D) \geq 2$;

(iv) $K_S + \ulcorner A \urcorner + D$ is effective if A is a nef and big $\mathbb{Q}$-divisor and if $h^0(S,D) \geq 2$.

Proof. Both (i) and (ii) are direct corollaries of [Rr, Corollary 2]. (iii) is derived by a simple use of Riemann-Roch. To prove (iv), we may suppose that $|D|$ is base point free. Denote by C a generic irreducible element of $|D|$, then the vanishing theorem gives the exact sequence

$$H^0(S, K_S + \ulcorner A \urcorner + C) \longrightarrow H^0(C, K_C + H) \longrightarrow 0,$$

where $H := \ulcorner A \urcorner|_C$ is a divisor of positive degree. It is obvious that $h^0(C, K_C + H) \geq 2$ since C is a curve of genus ≥ 2. The proof is completed. ■

Lemma 2.2. *Let X be a nonsingular projective variety of dimension d, $D \in \mathrm{Div}(X)\otimes\mathbb{Q}$ be a $\mathbb{Q}$-divisor on X. Then we have the following:*

(i) if S is a smooth irreducible divisor on X, then $\ulcorner D \urcorner|_S \geq \ulcorner D|_S \urcorner$;

(ii) if $\pi : X' \longrightarrow X$ is a birational morphism, then $\pi^*(\ulcorner D \urcorner) \geq \ulcorner \pi^*(D) \urcorner$.

Proof. These statements are obvious. One only has to verify for effective $\mathbb{Q}$-divisors. ■

Lemma 2.3. *Let S be a smooth projective surface of general type, A be a nef and big $\mathbb{Q}$-divisor on S and $L := \ulcorner A \urcorner$, D be a Cartier divisor with $h^0(S,D) \geq 2$. Suppose $K_S + L$ is effective, $\Phi_{|D|}$ is a morphism and $L \cdot C \geq 3$, where C is a generic irreducible element of the moving part of $|D|$. Then $\Phi_{|K_S+L+D|}$ is a birational map.*

Proof. For simplicity, we can suppose that $|D|$ is base point free. If $\dim\Phi_{|D|}(S) = 2$, by (P1), it is sufficient to prove that $\Phi_{|K_S+L+D|}|_C$ is birational since K_S+L is effective by assumption. If $|D|$ is composed of a pencil, we can write $D \sim_{\text{lin}} \sum C_i$. Using the Kawamata-Viehweg vanishing theorem, we can easily see that $\Phi_{|K_S+L+D|}$ can't only distinguish different general fibers of $\Phi_{|D|}$, but also distinguish disjoint components in a general fiber of $\Phi_{|D|}$. So, by (P2), it is also sufficient to verify the birationality of $\Phi_{|K_S+L+D|}|_C$. We have

$$|K_S + L + C| \;\Big|_C = |K_C + D|$$

by the vanishing theorem, where $D := L|_C$ is a divisor of degree ≥ 3. Thus $\Phi_{|K_C+D|}$ is an embedding and then the lemma is true. ■

Corollary 2.4. *Let S be a smooth projective surface of general type, A be a nef and big $\mathbb{Q}$-divisor on S and $L := \ulcorner A \urcorner$, D be a Cartier divisor with $h^0(S, D) \geq 2$, G is another Cartier divisor. Suppose* $\dim \Phi_{|G|}(C) = 1$ *where C is a generic irreducible element of the moving part of $|D|$. Then $\Phi_{|K_S+L+G+D|}$ is a birational map.*

Proof. One can suppose that both $|G|$ and $|D|$ are base point free. Then it is obvious that $G \cdot C \geq 2$. According to Lemma 2.1(iv), $K_S + L + G$ is effective. Since $A + G$ is nef and big and $(L + G) \cdot C \geq 3$, Lemma 2.3 directly derives the corollary. ■

Lemma 2.5. *Let X' be a smooth projective 3-fold of general type. Then*
(i) $P_2 \geq 4$ if $\chi(\mathcal{O}_{X'}) < 0$;
(ii) $P_4 \geq 3$ if $\chi(\mathcal{O}_{X'}) = 0$;
(iii) $P_{24} \geq 2$ if $\chi(\mathcal{O}_{X'}) = 1$.

Proof. These are Fletcher's results. One may refer to [F, 4.2, 4.4]. ■

Lemma 2.6. *Let X' be a smooth projective 3-fold of general type, $q(X') > 0$. Then*

$$P_{20}(X') \geq 2$$

with the possible exception of $q(X') = 2$, $\chi(\mathcal{O}_{X'}) = 0$.

Proof. This is an announcement of Kollár in [Ko, Remark 6.6]. ■

Lemma 2.7. *(*[Rr, Theorem 1]*) Let S be a smooth projective surface of general type, L be a nef divisor and $L^2 \geq 5$. Suppose p is a base point of $|K_S + L|$, then there exists an effective divisor E passing through p such that*

$$\text{either } \ L \cdot E = 0, \ E^2 = -1$$
$$\text{or } \ L \cdot E = 1, \ E^2 = 0.$$

Corollary 2.8. *Let S be a smooth minimal projective surface of general type, then*
(i) $|4K_S|$ is base point free.
(ii) $|3K_S|$ is base point free whenever $K_S^2 \geq 2$.

Proof. This is direct from Lemma 2.7. ■

3. Main Theorems

Recalling Definition 0.1, sometimes for simplicity, we denote $k_0(X)$ and $k_s(X)$ by k_0 and k_s, respectively.

Proposition 3.1. *Let X be a minimal projective 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. If $\dim \phi_{k_0}(X) \geq 2$, then $P_m(X) \geq 2$ for all $m \geq 2k_0$.*

Proof. First we take a birational modification $\pi : X' \longrightarrow X$, according to Hironaka, such that
(1) X' is smooth;
(2) $|k_0 K_{X'}|$ defines a morphism;
(3) the fractional part of $\pi^*(K_X)$ has supports with only normal crossings.

Denote by $S_0 := S_{k_0}$ the generic irreducible element of the moving part of $|k_0K_{X'}|$, then S_0 is a smooth projective surface of general type by Bertini's theorem. By the vanishing theorem, we have the exact sequence

$$H^0(X', K_{X'} + \ulcorner (t+k_0)\pi^*(K_X)\urcorner + S_0) \longrightarrow H^0(S_0, K_{S_0} + \ulcorner (t+k_0)\pi^*(K_X)\urcorner|_{S_0}) \longrightarrow 0,$$

where $t \geq 0$ is a given integer and

$$\ulcorner (t+k_0)\pi^*(K_X)\urcorner|_{S_0} \geq \ulcorner t\pi^*(K_X)|_{S_0}\urcorner + D_0,$$

$D := S_0|_{S_0}$ has the property $h^0(S_0, D) \geq 2$ according to the assumption. If $t = 0$, then

$$P_{2k_0+1}(X) \geq h^0(S_0, K_{S_0} + D) \geq 2$$

by Lemma 2.1(iii). If $t > 0$, we still have the following exact sequence

$$H^0(S_0, K_{S_0} + \ulcorner t\pi^*(K_X)|_{S_0}\urcorner + C) \longrightarrow H^0(K_C + G) \longrightarrow 0,$$

where C is a generic irreducible element of the moving part of $|D|$ and

$$G := \ulcorner t\pi^*(K_X)|_{S_0}\urcorner|_C$$

is a divisor of positive degree on C. Since C is a curve of genus ≥ 2, we have

$$h^0(C, K_C + G) \geq 2.$$

We can easily see that $P_{2k_0+t+1} \geq 2$. The proof is completed. ■

Corollary 3.2. *Let X be a minimal projective 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. If ϕ_{k_0} is birational, then $k_s \leq 3k_0$.*

Proof. This is obvious according to Proposition 3.1. ■

Theorem 3.3. *Let X be a minimal projective 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. If $\dim\phi_{k_0}(X) = 3$ and ϕ_{k_0} is not birational, then $k_s \leq 3k_0 + 2$.*

Proof. Taking the same modification $\pi : X' \longrightarrow X$ as in the proof of Proposition 3.1, we still denote by S_0 the general member of the moving part of $|k_0K_{X'}|$. Note that both $|k_0K_{X'}|$ and $|\ulcorner k_0\pi^*(K_X)\urcorner|$ have the same moving part. For a given integer $t > 0$, we have

$$K_{X'} + \ulcorner (t+2k_0)\pi^*(K_X)\urcorner + S_0 \leq (t + 3k_0 + 1)K_{X'}.$$

It is sufficient to prove the birationality of rational map given by

$$|K_{X'} + \ulcorner (t+2k_0)\pi^*(K_X)\urcorner + S_0|.$$

Because

$$K_{X'} + \ulcorner (t+2k_0)\pi^*(K_X)\urcorner$$

is effective according to Proposition 3.1, by virtue of (P1), we have to prove the birationality of

$$\Phi_{|K_{X'}+\ulcorner(t+2k_0)\pi^*(K_X)\urcorner+S_0|}\Big|_{S_0}.$$

We have the following exact sequence according to the vanishing theorem

$$H^0(X, K_{X'}+\ulcorner(t+2k_0)\pi^*(K_X)\urcorner+S_0)\longrightarrow H^0(S_0, K_{S_0}+\ulcorner(t+2k_0)\pi^*(K_X)\urcorner|_{S_0})\longrightarrow 0,$$

which means

$$|K_{X'}+\ulcorner(t+2k_0)\pi^*(K_X)\urcorner+S_0|\Big|_{S_0}=|K_{S_0}+\ulcorner(t+2k_0)\pi^*(K_X)\urcorner|_{S_0}|.$$

Noting that

$$K_{S_0}+\ulcorner(t+2k_0)\pi^*(K_X)\urcorner|_{S_0}\geq K_{S_0}+\ulcorner t\pi(K_X)|_{S_0}\urcorner+2L_0,$$

where $L_0:=S_0|_{S_0}$, we want to show that

$$\Phi_{|K_{S_0}+\ulcorner t\pi(K_X)|_{S_0}\urcorner+2L_0|}$$

is birational. Because $|L_0|$ gives a generically finite map, we see from Lemma 2.1(iv) that

$$K_{S_0}+\ulcorner t\pi^*(K_X)|_{S_0}\urcorner+L_0$$

is effective. On the other hand, let C be a generic irreducible element of $|L_0|$, then $\dim\Phi_{|L_0|}(C)=1$. Applying Corollary 2.4, we see that

$$|K_{S_0}+\ulcorner t\pi^*(K_X)|_{S_0}\urcorner+2L_0|$$

gives a birational map. The proof is completed. ■

Theorem 3.4. *Let X be a minimal projective 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. If $\dim\phi_{k_0}(X)=2$, then $k_s\leq 4k_0+4$.*

Proof. First we take the same modification $\pi: X'\longrightarrow X$ as in the proof of Proposition 3.1. We also suppose that S_0 is the moving part of $|k_0K_{X'}|$. For a given integer $t>0$, we obviously have

$$K_{X'}+\ulcorner(t+2k_0+2)\pi^*(K_X)\urcorner+2S_0\leq(t+4k_0+3)K_{X'}.$$

Thus it is sufficient to verify the birationality of the rational map given by

$$|K_{X'}+\ulcorner(t+2k_0+2)\pi^*(K_X)\urcorner+2S_0|.$$

By Proposition 3.1,

$$K_{X'}+\ulcorner(t+2k_0+2)\pi^*(K_X)\urcorner+S_0$$

is effective. According to (P1), we only have to prove the birationality of the restriction

$$\Phi_{|K_{X'}+\ulcorner(t+2k_0+2)\pi^*(K_X)\urcorner+2S_0|}\Big|_{S_0}$$

for the general S_0. The vanishing theorem gives the exact sequence

$$\begin{aligned}&H^0(X', K_{X'} + \ulcorner (t+2k_0+2)\pi^*(K_X)\urcorner + 2S_0)\\ \longrightarrow &H^0(S_0, K_{S_0} + \ulcorner (t+2k_0+2)\pi^*(K_X)\urcorner \big|_{S_0} + S_0|_{S_0}) \longrightarrow 0.\end{aligned}$$

This means

$$\Phi_{|K_{X'}+\ulcorner (t+2k_0+2)\pi^*(K_X)\urcorner+2S_0|}\big|_{S_0} = \Phi_{|K_{S_0}+\ulcorner (t+2k_0+2)\pi^*(K_X)\urcorner|_{S_0}+S_0|_{S_0}|}.$$

Suppose M_{2k_0+2} is the moving part of $|(2k_0+2)K_{X'}|$, we have to study some property of $\big| M_{2k_0+2}|_{S_0} \big|$. Note that M_{2k_0+2} is also the moving part of $|\ulcorner (2k_0+2)\pi^*(K_X)\urcorner|$. We have

$$K_{X'} + \ulcorner \pi^*(K_X)\urcorner + 2S_0 \leq (2k_0+2)K_{X'}.$$

The vanishing theorem gives the exact sequence

$$H^0(X', K_{X'} + \ulcorner \pi^*(K_X)\urcorner + 2S_0) \stackrel{\alpha}{\longrightarrow} H^0(S_0, K_{S_0} + \ulcorner \pi^*(K_X)\urcorner|_{S_0} + L_0) \longrightarrow 0,$$

where $L_0 := S_0|_{S_0}$. Denote by M'_{2k_0+2} the moving part of

$$|K_{X'} + \ulcorner \pi^*(K_X)\urcorner + 2S_0|$$

and by G the moving part of

$$|K_{S_0} + \ulcorner \pi^*(K_X)\urcorner|_{S_0} + L_0|.$$

Considering the natural map

$$H^0(X', M'_{2k_0+2}) \stackrel{\beta}{\longrightarrow} H^0(S_0, M'_{2k_0+2}|_{S_0}),$$

we have

$$\begin{aligned}&h^0(S_0, M'_{2k_0+2}|_{S_0}) \geq \dim_{\mathbb{C}}\big(\mathrm{im}(\beta)\big) = \dim_{\mathbb{C}}\big(\mathrm{im}(\alpha)\big)\\ =&h^0(S_0, K_{S_0} + \ulcorner \pi^*(K_X)\urcorner|_{S_0} + L_0).\end{aligned}$$

Because

$$M'_{2k_0+2}|_{S_0} \leq K_{S_0} + \ulcorner \pi^*(K_X)\urcorner|_{S_0} + L_0,$$

we see that

$$G \leq M'_{2k_0+2}|_{S_0} \leq M_{2k_0+2}|_{S_0}.$$

Noting that $|L_0|$ is a free pencil, we can suppose C is a generic irreducible element of $|L_0|$. Now the key step is to show that $\dim\Phi_{|G|}(C) = 1$. In fact, the vanishing theorem gives

$$|K_{S_0} + \ulcorner \pi^*(K_X)|_{S_0}\urcorner + L_0| \big|_C = |K_C + D|,$$

where $D := \ulcorner \pi^*(K_X)|_{S_0}\urcorner \big|_C$ is a divisor of positive degree. Because C is a curve of genus ≥ 2, $|K_C + D|$ gives a finite map. This shows

$$\dim\Phi_{|K_{S_0}+\ulcorner \pi^*(K_X)|_{S_0}\urcorner+L_0|}(C) = 1,$$

thus $\dim\Phi_{|G|}(C)=1$. Therefore

$$\dim\Phi_{|M_{2k_0+2}|_{S_0}|}(C)=1.$$

Noting that

$$h^0(S_0, M_{2k_0+2}|_{S_0}) \geq h^0(S_0, G) \geq 2,$$

we see from Lemma 2.1(iv) that

$$K_{S_0}+\ulcorner t\pi^*(K_X)|_{S_0}\urcorner+M_{2k_0+2}|_{S_0}$$

is effective. Finally, Lemma 2.3 gives the birationality of the rational map given by

$$|K_{S_0}+\ulcorner t\pi^*(K_X)|_{S_0}\urcorner+M_{2k_0+2}|_{S_0}+L_0|.$$

Because

$$\begin{aligned}&|K_{S_0}+\ulcorner t\pi^*(K_X)|_{S_0}\urcorner+M_{2k_0+2}|_{S_0}+L_0|\\ \subset&|K_{S_0}+\ulcorner(t+2k_0+2)\pi^*(K_X)\urcorner|_{S_0}+L_0|,\end{aligned}$$

so

$$\Phi_{|K_{S_0}+\ulcorner(t+2k_0+2)\pi^*(K_X)\urcorner|_{S_0}+S_0|_{S_0}|}$$

is birational. We have proved the theorem. ■

From now on, we suppose that $\dim\phi_{k_0}(X)=1$. This is the case which prevents us from getting a better bound for k_s. We can take the same modification $\pi: X'\longrightarrow X$ as in the proof of Proposition 3.1. Set $g:=\phi_{k_0}\circ\pi$ be the morphism from X' onto

$$W\subset\mathbb{P}^{P_{k_0}-1},$$

where W is the closed closure of the image of X through ϕ_{k_0}. Let

$$g: X'\xrightarrow{f} C\longrightarrow W$$

be the Stein-factorization, then C is a smooth projective curve. Denote $b:=g(C)$, the genus of C. If $b>0$, it is very easy to see by Kawamata's vanishing theorem for Weil divisors that $k_s\leq 2k_0+4$. (One may also refer to the proof of [Ch, Theorem 2.3.1].) In the rest of this section, we mainly study the case when C is the rational curve $\mathbb{P}^1$. We have a fibration $f: X'\longrightarrow\mathbb{P}^1$. Let S be a general fiber of the fibration, then S is a smooth projective surface of general type. Note that S is also the generic irreducible element of the moving part of the system $|k_0K_{X'}|$.

According to the behavior of the tricanonical map of S, we classify S into two types:

$(I)_t$ S is not a surface with $(K^2, p_g)=(1,2)$ and $(2,3)$, where the invariants represent the ones of the minimal model of S;

$(II)_t$ S is a surface with $(K^2,p_g)=(1,2)$ or $(2,3)$.

If S is of type $(I)_t$, then ϕ_{7k_0+3} is birational according to (1.5.2).

Theorem 3.5. *Let X be a minimal projective 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. If* $\dim\phi_{k_0}(X)=1$, $b=0$ *and S is of type $(II)_t$, then*

$$k_s \le 5k_0+5.$$

Proof. Because S is of type $(II)_t$, we always have $q(S)=0$ and $p_g(S)\ge 2$. We shall formulate our proof into steps and take S be a general fiber of f.

Step 1. $\dim\phi_{2k_0+1}(S)\ge 1$

Noting that

$$|K_{X'}+\ulcorner k_0\pi^*(K_X)\urcorner+\sum S_i|\subset|(2k_0+1)K_{X'}|,$$

and that the vanishing theorem gives

$$\begin{aligned}&|K_{X'}+\ulcorner k_0\pi^*(K_X)\urcorner+\sum S_i|\,\big|_S\\ =&\big|\,K_S+\ulcorner k_0\pi^*(K_X)\urcorner|_S\,\big|\supset|K_S|\end{aligned}$$

we obviously see that $\dim\phi_{2k_0+1}(S)\ge 1$, since $p_g(S)\ge 2$.

Step 2. $\dim\phi_{3k_0+2}(S)=2$

Noting that

$$|K_{X'}+\ulcorner(2k_0+1)\pi^*(K_X)\urcorner+\sum S_i|\subset|(3k_0+2)K_{X'}|,$$

and that the vanishing theorem gives

$$\begin{aligned}&|K_{X'}+\ulcorner(2k_0+1)\pi^*(K_X)\urcorner+\sum S_i|\,\big|_S\\ =&\big|\,K_S+\ulcorner(2k_0+1)\pi^*(K_X)\urcorner|_S\,\big|\\ \supset&\big|\,K_S+M_{2k_0+1}|_S\,\big|\end{aligned}$$

where M_{2k_0+1} is the moving part of $|\ulcorner(2k_0+1)\pi^*(K_X)\urcorner|$ and thus $h^0(S,M_{2k_0+1}|_S)\ge 2$ according to Step 1. Now it is sufficient to see that

$$\big|\,K_S+M_{2k_0+1}|_S\,\big|$$

gives a generically finite map, which is obvious because $q(S)=0$, $p_g(S)>0$ and S is of general type. In fact, one only has to study the restriction to a generic irreducible element of the moving part of $\big|\,M_{2k_0+1}|_S\,\big|$. Therefore $\dim\phi_{3k_0+2}(S)=2$.

Step 3. $mK_{X'}$ is effective whenever $m\ge 3k_0+2$

For a given integer $t>0$, we have

$$K_{X'}+\ulcorner(t+2k_0+1)\pi^*(K_X)\urcorner+\sum S_i\le(t+3k_0+2)K_{X'}.$$

The vanishing theorem gives

$$\begin{aligned}&|K_{X'}+\ulcorner(t+2k_0+1)\pi^*(K_X)\urcorner+\sum S_i|\,\big|_S\\ =&\big|\,K_S+\ulcorner(t+2k_0+1)\pi^*(K_X)\urcorner|_S\,\big|\\ \supset&\big|\,K_S+\ulcorner t\pi^*(K_X)\urcorner|_S+M_{2k_0+1}|_S\,\big|\end{aligned}$$

For simplicity, we can suppose that $\Phi_{|M_{2k_0+1}|_S|}$ is a morphism and denote by C a generic irreducible element of the moving part of the system $\big| M_{2k_0+1}|_S \big|$. The vanishing theorem also gives

$$\big| K_S + \ulcorner t\pi^*(K_X)|_S \urcorner + M_{2k_0+1}|_S \big|\Big|_C \supset |K_C + D|,$$

where $D := \ulcorner t\pi^*(K_X)|_S \urcorner|_C$ is a divisor of positive degree. Because C is a curve of genus ≥ 2, we see that $h^0(C, K_C + D) \geq 2$. The proof is completed.

Step 4. Studying of $|(4k_0+3)K_{X'}|$

This is an important step of our technique. First we have

$$|K_{X'} + \ulcorner (3k_0+2)\pi^*(K_X) \urcorner + \sum S_i| \subset |(4k_0+3)K_{X'}|.$$

Denote by M_{3k_0+2}, M_{4k_0+3} the moving part of

$$|(3k_0+2)K_{X'}|, \quad |(4k_0+3)K_{X'}|$$

respectively. Also denote by M'_{4k_0+3} the moving part of the system

$$|K_{X'} + \ulcorner (3k_0+2)\pi^*(K_X) \urcorner + \sum S_i|.$$

The vanishing theorem gives the following exact sequence

$$\begin{aligned} &H^0(X', K_{X'} + \ulcorner (3k_0+2)\pi^*(K_X) \urcorner + \sum S_i) \\ \xrightarrow{\alpha_1} &H^0(S, K_S + \ulcorner (3k_0+2)\pi^*(K_X) \urcorner|_S) \longrightarrow 0. \end{aligned}$$

We also have a natural map

$$H^0(X', M'_{4k_0+3}) \xrightarrow{\beta_1} H^0(S, M'_{4k_0+3}|_S).$$

From these maps, we can see that

$$\begin{aligned} &h^0(S, M'_{4k_0+3}) \geq \dim_{\mathbb{C}}\big(\mathrm{im}(\beta_1)\big) = \dim_{\mathbb{C}}\big(\mathrm{im}(\alpha_1)\big) \\ =&h^0(S, K_S + \ulcorner (3k_0+2)\pi^*(K_X) \urcorner|_S). \end{aligned}$$

Denote by G' the moving part of $\big| K_S + \ulcorner (3k_0+2)\pi^*(K_X) \urcorner|_S \big|$. Since

$$M'_{4k_0+3} \leq K_S + \ulcorner (3k_0+2)\pi^*(K_X) \urcorner|_S,$$

we see that $G' \leq M'_{4k_0+3}|_S$. Denote by G_0, G the moving parts of

$$|K_S|, \quad \big| \ulcorner (3k_0+2)\pi^*(K_X) \urcorner|_S \big|$$

respectively. Then $G' \geq G_0 + G$ and thus

$$G_0 + G \leq M_{4k_0+3}|_S.$$

Furthermore, we should have $h^0(S, G_0) \geq 2$ and $\dim\Phi_{|G|}(S) = 2$. If C is a generic irreducible element of $|G_0|$, then $\dim\Phi_{|G|}(C) = 1$.

Step 5. The birationality

For a given integer $t > 0$, we study the system

$$|K_{X'} + \ulcorner(t + 4k_0 + 3)\pi^*(K_X)\urcorner + \sum S_i|.$$

According to Step 3,

$$K_{X'} + \ulcorner(t + 4k_0 + 3)\pi^*(K_X)\urcorner$$

is effective. In order to use (P1), it is enough to study the restriction. The vanishing theorem gives

$$\begin{aligned} &|K_{X'} + \ulcorner(t + 4k_0 + 3)\pi^*(K_X)\urcorner + \sum S_i| \ |_S \\ =& \ \big|\ K_S + \ulcorner(t + 4k_0 + 3)\pi^*(K_X)\urcorner|_S\ \big| \\ \supset& \ \big|\ K_S + \ulcorner t\pi^*(K_X)|_S\urcorner + G + G_0\ \big|\ . \end{aligned}$$

By Corollary 2.4 and Step 4, we see that

$$\big|\ K_S + \ulcorner t\pi^*(K_X)|_S\urcorner + G + G_0\ \big|$$

gives a birational map. The theorem has been proved. ■

Corollary 3.6. *Let X be a minimal projective 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. Then either ϕ_{7k_0+3} is birational or $k_s \leq 5k_0 + 5$. In particular, ϕ_{7k_0+3} is definitely birational.*

Proof. This is a direct result from Theorem 3.3, Theorem 3.4, Theorem 3.5 and (1.5.2). ■

In order to prove the stable birationality, we need to classify surfaces into the following 3 types, where we suppose S is a smooth projective surface of general type:
$(I)_s$ $p_g(S) \geq 2$;
$(II)_s$ $p_g(S) \leq 1$ and $K_{S_0}^2 \geq 2$, where S_0 is the minimal model of S;
$(III)_s$ $p_g(S) \leq 1$ and $K_{S_0}^2 = 1$.

Theorem 3.7. *Let X be a minimal projective 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. If $\dim\phi_{k_0}(X) = 1$, $b = 0$ and S is of type $(I)_s$, then*

$$k_s \leq 6k_0 + 5.$$

Proof. It is obvious that type $(II)_t$ is a special one of type $(I)_s$. However, one may use a similar argument to that of Theorem 3.5. One point to note here is that S may be not only regular but also irregular. So the bound for k_s is slightly weaker than in Theorem 3.5. We keep the same notations as in the proof of Theorem 3.5. We shall omit unnecessary redundancy by virtue of the argument there. Suppose S is a general fiber of the derived fibration $f : X' \longrightarrow \mathbb{P}^1$.

Step 1. $\dim\phi_{2k_0+1}(S)=1$. (omitted)

Step 2. $\dim\phi_{4k_0+2}(S)=2$.

Noting that

$$|K_{X'}+\ulcorner(3k_0+1)\pi^*(K_X)\urcorner+\sum S_i|\subset|(4k_0+2)K_{X'}|,$$

and that the vanishing theorem gives

$$\begin{aligned}&|K_{X'}+\ulcorner(3k_0+1)\pi^*(K_X)\urcorner+\sum S_i|\ \big|_S\\ =&\ \big|\ K_S+\ulcorner(3k_0+1)\pi^*(K_X)\urcorner|_S\ \big|\\ \supset&\ \big|\ K_S+\ulcorner k_0\pi^*(K_X)|_S\urcorner+M_{2k_0+1}|_S\ \big|\end{aligned}$$

where M_{2k_0+1} is the moving part of $|\ulcorner(2k_0+1)\pi^*(K_X)\urcorner|$ and thus $h^0(S,M_{2k_0+1}|_S)\geq 2$ according to Step 1. Now it is sufficient to see that

$$\big|\ K_S+\ulcorner k_0\pi^*(K_X)|_S\urcorner+M_{2k_0+1}|_S\ \big|$$

gives a generically finite map. In fact, $K_S+\ulcorner k_0\pi^*(K_X)|_S\urcorner$ is effective, $k_0\pi^*(K_X)|_S$ is nef and big, and for the generic irreducible element C of the moving part of $\big|\ M_{2k_0+1}|_S\ \big|$, it is easy to see that

$$\Phi_{|K_S+\ulcorner k_0\pi^*(K_X)|_S\urcorner+M_{2k_0+1}|_S|}$$

can distinguish different generic irreducible elements $C's$. The vanishing theorem gives

$$\big|\ K_S+\ulcorner k_0\pi^*(K_X)|_S\urcorner+C\ \big|\big|_C=|K_C+D|,$$

where $D:=\ulcorner k_0\pi^*(K_X)|_S\urcorner|_C$ is a divisor of positive degree. Thus $|K_C+D|$ gives a finite map, and so does

$$\big|\ K_S+\ulcorner k_0\pi^*(K_X)|_S\urcorner+M_{2k_0+1}|_S\ \big|$$

bu virtue of (P2). Therefore $\dim\phi_{4k_0+2}(S)=2$.

Step 3. $mK_{X'}$ is effective whenever $m\geq 3k_0+2$. (omitted)

Step 4. Studying of $|(5k_0+3)K_{X'}|$.

First we have

$$|K_{X'}+\ulcorner(4k_0+2)\pi^*(K_X)\urcorner+\sum S_i|\subset|(5k_0+3)K_{X'}|.$$

Denote by M_{4k_0+2}, M_{5k_0+3} the moving part of

$$|(4k_0+2)K_{X'}|,\quad |(5k_0+3)K_{X'}|$$

respectively. Also denote by M'_{5k_0+3} the moving part of the system

$$|K_{X'}+\ulcorner(4k_0+2)\pi^*(K_X)\urcorner+\sum S_i|.$$

The vanishing theorem gives the following exact sequence

$$\begin{aligned}&H^0(X', K_{X'} + \ulcorner(4k_0+2)\pi^*(K_X)\urcorner + \sum S_i)\\ \xrightarrow{\alpha_1}&H^0(S, K_S + \ulcorner(4k_0+2)\pi^*(K_X)\urcorner|_S) \longrightarrow 0.\end{aligned}$$

We also have a natural map

$$H^0(X', M'_{5k_0+3}) \xrightarrow{\beta_1} H^0(S, M'_{5k_0+3}|_S).$$

From these maps, we can see that

$$\begin{aligned}&h^0(S, M'_{5k_0+3}) \geq \dim_{\mathbb{C}}\big(\mathrm{im}(\beta_1)\big) = \dim_{\mathbb{C}}\big(\mathrm{im}(\alpha_1)\big)\\ =&h^0(S, K_S + \ulcorner(4k_0+2)\pi^*(K_X)\urcorner|_S).\end{aligned}$$

Denote by G'' the moving part of $\big|\ K_S + \ulcorner(4k_0+2)\pi^*(K_X)\urcorner|_S\ \big|$. Since

$$M'_{5k_0+3}|_S \leq K_S + \ulcorner(4k_0+2)\pi^*(K_X)\urcorner|_S,$$

we see that $G'' \leq M'_{5k_0+3}|_S$. Denote by G_0, G the moving parts of

$$|K_S|, \quad \big|\ \ulcorner(4k_0+2)\pi^*(K_X)\urcorner|_S\ \big|$$

respectively. Then $G'' \geq G_0 + G$ and thus

$$G_0 + G \leq M_{5k_0+3}|_S.$$

Furthermore, we should have $h^0(S, G_0) \geq 2$ and $\dim\Phi_{|G|}(S) = 2$. If $\overline{C}$ is a generic irreducible element of $|G_0|$, then $\dim\Phi_{|G|}(\overline{C}) = 1$.

Step 5. The birationality

For a given integer $t > 0$, we study the system

$$|K_{X'} + \ulcorner(t+5k_0+3)\pi^*(K_X)\urcorner + \sum S_i|.$$

According to Step 3,

$$K_{X'} + \ulcorner(t+5k_0+3)\pi^*(K_X)\urcorner$$

is effective. In order to use (P2), it is enough to study the restriction to S. The vanishing theorem gives

$$\begin{aligned}&|K_{X'} + \ulcorner(t+5k_0+3)\pi^*(K_X)\urcorner + \sum S_i|\ \big|_S\\ =&\ \big|\ K_S + \ulcorner(t+5k_0+3)\pi^*(K_X)\urcorner|_S\ \big|\\ \supset&\ \big|\ K_S + \ulcorner t\pi^*(K_X)|_S\urcorner + G + G_0\ \big|\ .\end{aligned}$$

By Lemma 2.4 and Step 4, we see that

$$\big|\ K_S + \ulcorner t\pi^*(K_X)|_S\urcorner + G + G_0\ \big|$$

gives a birational map. The theorem has been proved. ■

Proposition 3.8. *Let X be a minimal projective 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. If* $\dim\phi_{k_0}(X)=1$, $b=0$ *and S is of type $(II)_s$ or $(III)_s$, then $mK_{X'}$ is effective whenever $m\geq 6k_0+3$.*

Proof. According to (1.5.1), $\dim\phi_{5k_0+2}(X)\geq 2$. For a given integer $t\geq 0$, we want to study the system

$$|K_{X'}+\ulcorner(t+5k_0+2)\pi^*(K_X)\urcorner+\sum S_i|.$$

Now using a parallel argument to that of Step 3 in the proof of Theorem 3.5, one can easily get the result. ■

Theorem 3.9. *Let X be a minimal projective 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. If* $\dim\phi_{k_0}(X)=1$, $b=0$ *and S is of type $(II)_s$, then*

$$k_s\leq 9k_0+6.$$

Proof. Since S is of type $(II)_s$, the technique of Theorem 3.5 is not effective here. We shall study in an alternative way. The key step is to study the system $|(7k_0+3)K_{X'}|$. Denote by M_{7k_0+3} the moving part of $|(7k_0+3)K_{X'}|$. It is obvious that

$$\Phi_{|(7k_0+3)K_{X'}|}=\Phi_{|M_{7k_0+3}|}.$$

For a general fiber S, we suppose N_3 is the moving part of $|3K_S|$. By virtue of Kollár's technique, we know that the global sections of $|3K_S|$ extends to global sections of

$$|(7k_0+3)K_{X'}|$$

and so that

$$\Phi_{|(7k_0+3)K_{X'}|}|_S$$

does behave more than $\Phi_{|3K_S|}$. This means we should have

$$|M_{7k_0+3}|\ \big|_S\supset|N_3|.$$

Now let

$$\sigma:S\longrightarrow S_0$$

be the natural contraction onto the minimal model S_0. By Lemma 2.8(ii), we know that $|3K_{S_0}|$ is base point free. So $\sigma^*(3K_{S_0})$ is linearly equivalent to the moving part N_3 of $|3K_S|$. We can write

$$N_3\sim_{\rm lin}\sigma^*(K_{S_0})+\sigma^*(2K_{S_0}).$$

According to [X], we know that $|\sigma^*(2K_{S_0})|$ defines a generically finite map.

The next step is to study $|(8k_0+4)K_{X'}|$. Denote by M_{8k_0+4} the moving part of

$$|(8k_0+4)K_{X'}|,$$

and by M'_{8k_0+4} the moving part of the system

$$|K_{X'}+\ulcorner(7k_0+3)\pi^*(K_X)\urcorner+\sum S_i|.$$

The vanishing theorem gives the exact sequence

$$\begin{aligned}&H^0(X', K_{X'} + \ulcorner(7k_0+3)\pi^*(K_X)\urcorner + \sum S_i)\\ \xrightarrow{\alpha_1'}&H^0(S, K_S + \ulcorner(7k_0+3)\pi^*(K_X)\urcorner|_S) \longrightarrow 0.\end{aligned}$$

We have another natural map

$$H^0(X', M'_{8k_0+4}) \xrightarrow{\beta_1'} H^0(S, M'_{8k_0+4}|_S).$$

It is obvious that

$$M'_{8k_0+4}|_S \leq K_S + \ulcorner(7k_0+3)\pi^*(K_X)\urcorner|_S.$$

On the other hand, we have

$$\begin{aligned}&h^0(S, M'_{8k_0+4}|_S) \geq \dim_{\mathbb{C}}\big(\mathrm{im}(\beta_1')\big) = \dim_{\mathbb{C}}\big(\mathrm{im}(\alpha_1')\big)\\ =&h^0(S, K_S + \ulcorner(7k_0+3)\pi^*(K_X)\urcorner|_S).\end{aligned}$$

So we have

$$G \leq M'_{8k_0+4}|_S \leq M_{8k_0+4}|_S,$$

where we denote by G the moving part of

$$\Big|\ K_S + \ulcorner(7k_0+3)\pi^*(K_X)\urcorner|_S\ \Big|\ .$$

Because

$$\begin{aligned}K_S + \ulcorner(7k_0+3)\pi^*(K_X)\urcorner|_S &\geq K_S + N_3\\ &= K_S + \sigma^*(K_{S_0}) + \sigma^*(2K_{S_0})\\ &\geq \sigma^*(2K_{S_0}) + \sigma^*(2K_{S_0})\\ &\geq N_2 + N_2,\end{aligned}$$

where N_2 is the moving part of the system $|\sigma^*(2K_{S_0})|$. Denote by C the generic irreducible element of $|N_2|$, then $\dim\Phi_{|N_2|}(C) = 1$.

For a given integer $t > 0$, we want to study the system

$$|K_{X'} + \ulcorner(t + 8k_0 + 4)\pi^*(K_X)\urcorner + \sum S_i|.$$

By Proposition 3.8, $K_{X'} + \ulcorner(t + 8k_0 + 4)\pi^*(K_X)\urcorner$ is effective. On the other hand, the vanishing theorem gives

$$\begin{aligned}&|K_{X'} + \ulcorner(t + 8k_0 + 4)\pi^*(K_X)\urcorner + \sum S_i|\ \big|_S\\ =&\Big|\ K_S + \ulcorner(t + 8k_0 + 4)\pi^*(K_X)\urcorner|_S\ \Big|\\ \supset&\Big|\ K_S + \ulcorner t\pi^*(K_X)|_S\urcorner + N_2 + N_2\ \Big|\ .\end{aligned}$$

Corollary 2.4 derives that

$$\Big|\ K_S + \ulcorner t\pi^*(K_X)|_S\urcorner + N_2 + N_2\ \Big|$$

defines a birational map. Therefore we see that ϕ_{t+9k_0+5} is birational for all $t > 0$. The proof is completed. ■

Theorem 3.10. *Let X be a minimal projective 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. If* $\dim\phi_{k_0}(X)=1$, $b=0$ *and S is of type* $(III)_s$, *then*

$$k_s \le 10k_0+6.$$

Proof. The technique is similar to the one in the last theorem. The key step is to study $|(9k_0+4)K_{X'}|$. Denote by M_{9k_0+4} the moving part of $|(9k_0+4)K_{X'}|$. It is obvious that

$$\Phi_{|(9k_0+4)K_{X'}|}=\Phi_{|M_{9k_0+4}|}.$$

For a general fiber S, we suppose N_4 is the moving part of $|4K_S|$. By virtue of Kollár's technique, we know that the global sections of $|4K_S|$ extends to global sections of

$$|(9k_0+4)K_{X'}|$$

and so that

$$\Phi_{|(9k_0+4)K_{X'}|}|_S$$

does behave more than $\Phi_{|4K_S|}$. This means that we should have

$$|M_{9k_0+4}|\ \big|_S \supset |N_4|.$$

Now let

$$\sigma : S \longrightarrow S_0$$

be the natural contraction onto the minimal model S_0. By Lemma 2.8(i), we know that $|4K_{S_0}|$ is base point free. So $\sigma^*(4K_{S_0})$ is linearly equivalent to the moving part N_4 of $|4K_S|$. We can write

$$\begin{aligned} N_4 &\sim_{\text{lin}} \sigma^*(2K_{S_0})+\sigma^*(2K_{S_0})\\ &\ge \sigma^*(2K_{S_0})+N_2 \end{aligned}$$

According to [X], we know that

$$\dim\Phi_{|\sigma^*(2K_{S_0})|}(S)\ge 1.$$

So $h^0(S,N_2)\ge 2$.

For a given integer $t>0$, we want to study the system

$$|K_{X'}+\ulcorner(t+9k_0+4)\pi^*(K_X)\urcorner+\sum S_i|.$$

By Proposition 3.8, $K_{X'}+\ulcorner(t+9k_0+4)\pi^*(K_X)\urcorner$ is effective. On the other hand, the vanishing theorem gives

$$\begin{aligned} &|K_{X'}+\ulcorner(t+9k_0+4)\pi^*(K_X)\urcorner+\sum S_i|\ \big|_S\\ =&\ \big|\ K_S+\ulcorner(t+9k_0+4)\pi^*(K_X)\urcorner|_S\ \big|\\ \supset&\ \big|\ K_S+\ulcorner t\pi^*(K_X)|_S\urcorner+\sigma^*(2K_{S_0})+N_2\ \big|\ . \end{aligned}$$

Lemma 2.1(iv) tells that

$$K_S+\ulcorner t\pi^*(K_X)|_S\urcorner+\sigma^*(2K_{S_0})$$

is effective, since $h^0(S,\sigma^*(2K_{S_0}))\ge 2$. Now using Lemma 2.3, we see that

$$\big|\ K_S+\ulcorner t\pi^*(K_X)|_S\urcorner+\sigma^*(2K_{S_0})+N_2\ \big|$$

defines a birational map. Therefore we see that ϕ_{t+10k_0+5} is birational for all $t>0$. The proof is completed. ■

4. Further discussion

From arguments of the last section, we have seen that the worse case possibly happens when $|k_0K_X|$ is composed of a rational pencil of surfaces with small invariants. Here, we go on studying this case in a more delicate way. We suppose $f : X' \longrightarrow \mathbb{P}^1$ is the derived fibration from $|k_0K_{X'}|$ and keep the same notations as in the previous section. From the spectral sequence:

$$E_2^{p,q} := H^p(\mathbb{P}^1, R^q f_*\omega_{X'}) \Rightarrow E^n := H^n(X', \omega_{X'}),$$

we get by direct calculation that

$$h^2(X', \mathcal{O}_{X'}) = h^1(\mathbb{P}^1, f_*\omega_{X'}) + h^0(\mathbb{P}^1, R^1 f_*\omega_{X'}),$$

$$q(X') := h^1(X', \mathcal{O}_{X'}) = h^1(\mathbb{P}^1, R^1 f_*\omega_{X'}).$$

Lemma 4.1. *(*[Ci, Theorem 3.1]*) Let S be a smooth projective minimal surface of general type, $p_g(S) \geq 1$, then $|2K_S|$ is base point free.*

The following lemma, as well as the proof, was provided by Prof. C. Ciliberto.

Lemma 4.2. *Let S be a smooth projective minimal surface of general type with $K_S^2 = 1$ and $p_g(S) = 1$, then $|3K_S|$ is base point free.*

Proof. Since $p_g(S) = 1$, we have only one canonical curve C. Because $q(S) = 0$, the line bundle $\mathcal{O}_C(K)$ has no global section, i.e. $h^0(C, \mathcal{O}_C(K)) = 0$. Let x be a base point of $|3K_S|$, then $x \in C$ since $|2K_S|$ is base point free according to Lemma 4.1. Considering the divisor $D = 2C \in |2K_S|$ and using Theorem 4.5 of [Ci] to the system $|K_S + D|$, we see that $D = A + B$, $A \cdot B = 1$. This leads to $A^2 + B^2 = 2$, $A^2 + 1 = 2K_S \cdot A \geq 0$ and $B^2 + 1 = 2K_S \cdot B \geq 0$. One can see from the Hodge Index Theorem that the only possibility is

$$A^2 = B^2 = K_S \cdot A = K_S \cdot B = 1.$$

Therefore it is easy to see that

$$A \sim_{\text{num}} B \sim_{\text{num}} C.$$

According to Bombieri ([Bo]), Pic(S) has no torsion element. Thus $A = B = C$. So Mendes Lopes Lemma (Theorem 4.5 of [Ci]) implies that x is a smooth point of C and

$$\mathcal{O}_C(x) = \mathcal{O}_C(C) = \mathcal{O}_C(K_S),$$

a contradiction. ■

Proposition 4.3. *Let X be a minimal projective 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. If $\dim\phi_{k_0}(X) = 1$, $b = 0$ and $p_g(S) = 1$, then*

$$k_s \leq 8k_0 + 6.$$

Proof. We know that $|3K_S|$ is base point free by virtue of both Corollary 2.8 and Lemma 4.2 and that $|2K_S|$ is base point free by Lemma 4.1. We are going to formulate the proof through steps.

Step 1. Studying of $|(5k_0+2)K_{X'}|$.

Denote by M_{5k_0+2} the moving part of $|(5k_0+2)K_{X'}|$. By virtue of Kollár's technique, we know that the global sections of $|2K_S|$ extends to global sections of

$$|(5k_0+2)K_{X'}|$$

and so that

$$\Phi_{|(5k_0+2)K_{X'}|}|_S$$

does behave more than $\Phi_{|2K_S|}$. This means that we should have

$$|M_{5k_0+2}|\ \big|_S \supset |N_2|,$$

where N_2 is the moving part of $|2K_S|$. Now let

$$\sigma : S \longrightarrow S_0$$

be the natural contraction onto the minimal model S_0. It is obvious that $N_2 = \sigma^*(2K_{S_0})$. So we get

$$|M_{5k_0+2}|\ \big|_S \supset |\sigma^*(2K_{S_0})|.$$

Step 2. Studying of $|(6k_0+3)K_{X'}|$.

Denote by M_{6k_0+3} the moving part of $|(6k_0+3)K_{X'}|$. The vanishing theorem gives

$$\begin{aligned} &|K_{X'} + \ulcorner(5k_0+2)\pi^*(K_X)\urcorner + S|\ \big|_S \\ =&\ \big|\ K_S + \ulcorner(5k_0+2)\pi^*(K_X)\urcorner|_S\ \big| \\ \supset&\ \big|\ K_S + M_{5k_0+2}|_S\ \big| \\ \supset&|K_S + \sigma^*(2K_{S_0})|. \end{aligned}$$

Suppose M'_{6k_0+3} is the moving part of the system

$$|K_{X'} + \ulcorner(5k_0+2)\pi^*(K_X)\urcorner + S|,$$

then it is not difficult to see that

$$|M'_{6k_0+3}|\ \big|_S \supset |N_3|,$$

where $N_3 = \sigma^*(3K_{S_0})$ is the moving part of

$$|K_S + \sigma^*(2K_{S_0})|.$$

So it is also true that

$$|M_{6k_0+3}|\ \big|_S \supset |\sigma^*(3K_{S_0})|.$$

Step 3. Studying of $|(7k_0+4)K_{X'}|$.

Denote by M_{7k_0+4} the moving part of the system $|(7k_0+4)K_{X'}|$. It is clear that

$$|K_{X'} + \ulcorner (6k_0+3)\pi^*(K_X)\urcorner + S| \subset |(7k_0+4)K_{X'}|.$$

The vanishing theorem gives

$$\begin{aligned} &|K_{X'} + \ulcorner (6k_0+3)\pi^*(K_X)\urcorner + S| \,\big|_S \supset \big|\ K_S + M_{6k_0+3}|_S\ \big| \\ \supset &|K_S + \sigma^*(3K_{S_0})| \supset |\sigma^*(2K_{S_0}) + \sigma^*(2K_{S_0})|. \end{aligned}$$

Denote by M'_{7k_0+4} the moving part of

$$|K_{X'} + \ulcorner (6k_0+3)\pi^*(K_X)\urcorner + S|,$$

then it is easy to see that

$$|M'_{7k_0+4}| \,\big|_S \supset |\sigma^*(2K_{S_0}) + \sigma^*(2K_{S_0})|.$$

So we should have

$$|M_{7k_0+4}| \,\big|_S \supset |\sigma^*(2K_{S_0}) + \sigma^*(2K_{S_0})|,$$

since $|\sigma^*(4K_{S_0})|$ is base point free.

Step 4. The stable birationality of ϕ_{8k_0+6}.

Given a positive integer $t > 0$, it is obvious that

$$|K_{X'} + \ulcorner (t+7k_0+4)\pi^*(K_X)\urcorner + S| \subset |(t+7k_0+5)K_{X'}|.$$

By Proposition 3.8, we see that

$$K_{X'} + \ulcorner (t+7k_0+4)\pi^*(K_X)\urcorner$$

is effective. In order to use (P2), it is sufficient to prove that

$$\Phi_{|K_{X'}+\ulcorner (t+7k_0+4)\pi^*(K_X)\urcorner+S|}|_S$$

is birational. The vanishing theorem gives

$$\begin{aligned} &|K_{X'} + \ulcorner (t+7k_0+4)\pi^*(K_X)\urcorner + S| \,\big|_S \\ = &\big|\ K_S + \ulcorner (t+7k_0+4)\pi^*(K_X)\urcorner|_S\ \big| \\ \supset &\big|\ K_S + \ulcorner t\pi^*(K_X)|_S\urcorner + \sigma^*(2K_{S_0}) + \sigma^*(2K_{S_0})\ \big|\,. \end{aligned}$$

Since $|\sigma^*(2K_{S_0})|$ gives a finite morphism, it is easy to see by Corollary 2.4 that

$$\big|\ K_S + \ulcorner t\pi^*(K_X)|_S\urcorner + \sigma^*(2K_{S_0}) + \sigma^*(2K_{S_0})\ \big|$$

gives a birational map. The proof is completed. ■

Corollary 4.4. *Let X be a minimal projective 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. If $k_0 \geq 25$, then*

$$k_s \leq 8k_0 + 6.$$

Proof. If $\dim\phi_{k_0}(X) \geq 2$ or $\dim\phi_{k_0}(X) = 1$ and $b > 0$, we have seen from the last section that $k_s \ll 8k_0 + 6$. If $\dim\phi_{k_0}(X) = 1$, $b = 0$ and S is of type $(I)_s$, one has $k_s \leq 6k_0 + 5$ according to Theorem 3.7. If $p_g(S) = 1$, then Proposition 4.3 implies $k_s \leq 8k_0 + 6$. The remain case is the one when $p_g(S) = 0$. We automatically have $q(S) = 0$. So, $q(X') = h^2(\mathcal{O}_{X'}) = 0$ and $\chi(\mathcal{O}_{X'}) = 1$. Lemma 2.5(iii) implies $k_0 \leq 24$. So if $k_0 \geq 25$, then the final case doesn't occur. ■

Corollary 4.5. *Let X be a minimal projective 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. If $q(X) := h^1(\mathcal{O}_X) > 0$, then ϕ_{166} is stably birational.*

Proof. For the same reason, we can suppose that $\dim\phi_{k_0}(X) = 1$ and $b = 0$. If $q(X') > 0$, then we should have $q(S) > 0$. So $p_g(S) > 0$. Using Proposition 4.3, we have $k_s \leq 8k_0 + 6$. Now according to both Lemma 2.5 and Lemma 2.6, we have $k_0 \leq 20$. So ϕ_{166} is stably birational. ■

Corollary 4.6. *Let X be a minimal projective 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. Suppose $q(X) > 1$ or $q(X) = 1$ but $\chi(\mathcal{O}_X) \neq 1$, then ϕ_{125} is stably birational.*

Proof. If S is of type $(I)_s$, then $k_s \leq 6k_0 + 5$. This means ϕ_{125} is stably birational, since $k_0 \leq 20$.

If S is of type $(II)_s$, then

$$q(S) \leq p_g(S) \leq 1.$$

Because $q(X') > 0$, we see that $q(S) > 0$. So we should have

$$q(X') = q(S) = p_g(S) = 1 \text{ and } R^1 f_* \omega_{X'} \cong \omega_{\mathbb{P}^1}.$$

Therefore

$$h^2(\mathcal{O}_{X'}) = h^1(\mathbb{P}^1, f_*\omega_{X'}) \leq 1.$$

Now we have

$$\chi(\mathcal{O}_{X'}) = 1 - q(X') + h^2(\mathcal{O}_{X'}) - p_g(X') \leq 1.$$

By assumption, $\chi(\mathcal{O}_{X'}) \neq 1$, so $\chi(\mathcal{O}_X) \leq 0$. Thus $k_0 \leq 4$ by Lemma 2.5. This means ϕ_{41} is stably birational according to Theorem 3.9. ■

Finally, recalling Definition 0.1, we would like to put forward the following

Conjecture. $\mu_s(3) \leq 6$.

This paper has proved that $\mu_s(3) \leq 13$. We know that $\mu_s(1) = 3$ and $\mu_s(2) = 5$ ([BPV]). For every minimal smooth projective 3-fold X of general type, it is true that $\mu_s(X) \leq 6$. No counter-examples have been found such that $\mu_s(X) > 6$. Recently, we were informed of a new example by Professor E. Stagnaro who constructed a smooth projective 3-fold Y of general type with

$$p_g(Y) = q(Y) = h^2(\mathcal{O}_Y) = 0, \ P_2 = 1, \ P_3 = 2$$

and ϕ_m is birational if and only if $m \geq 11$. So it is clear this example has the property $\mu_s(Y) = \frac{11}{3}$.

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References

[BPV] W. Barth, C. Peter, A. Van de Ven, *Compact Complex Surface*, Springer-Verlag 1984.

[Bo] E. Bombieri, *Canonical models of surfaces of general type*, Publ. Math. Inst. Hautes Etud. Sci. **42**(1973), 171-219.

[B-C] E. Bombieri, F. Catanese, *The tricanonical map of a surface with $K^2 = 2$, $p_g = 0$*, in C.P. Ramanujam, a tribute, Springer Verlag, Heidelberg (1978), 279-290.

[Ca] F. Catanese, *Pluricanonical mapping of surfaces with $K^2 = 1,\ 2$, $q = p_g = 0$*, C.I.M.E. Algebraic Surfaces, Liguori, Napoli(1981), 249-266.

[Ch] M. Chen, *The relative pluricanonical stability for 3-folds of general type*, Proc. Amer. Math. Soc. (to appear). math.AG/9902047

[Ci] C. Ciliberto, *The bicanonical map for surfaces of general type*, Proc. Symp. in Pure Math. **62**(1997), 57-83.

[D] O. Debarre, *Inégalités numériques pour les surfaces de type général*, Bull. Soc. Math. Francs **110**(1982), 319-346.

[E-L] L. Ein, R. Lazarsfeld, *Global generation of pluricanonical and adjoint linear systems on smooth projective threefolds*, J. Amer. Math. Soc. **6**(1993), 875-903.

[F] A. R. Fletcher, *Contributions to Riemann-Roch in projective 3-folds with only canonical singularities and application*, Proc. Sympos. Pure Math. **46**, Amer. Math. Soc. Providence (1987), 221-232.

[Ha] R. Hartshorne, Algebraic Geometry, GTM **52**, Springer-Verlag 1977.

[Hi] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero,* Ann. of Math. **79**(1964), 109-326.

[Ka] Y. Kawamata, *A generalization of Kodaira-Ramanujam's vanishing theorem*, Math. Ann., **261**(1982), 43-46.

[KMM] Y. Kawamata, K. Matsuda, K. Matsuki, *Introduction to the minimal model problem*, Adv. Stud. Pure Math. **10** (1987), 283-360.

[Ko] J. Kollár, *Higher direct images of dualizing sheaves I*, Ann. of Math. **123**(1986), 11-42.

[K-M] J. Kollár, S. Mori, *Birational geometry of algebraic varieties*, Cambridge Univ. Press, 1998.

[L] T. Luo, *Global 2-forms on regular 3-folds of general type*, Duke Math. J. **71**(1993), 859-869.

[Ma] K. Matsuki, *On pluricanonical maps for 3-folds of general type*, J. Math. Soc. Japan **38**(1986), 339-359.

[Mi] Y. Miyaoka, *Tricanonical maps of numerical Godeaux surfaces*, Invent. Math. **34** (1976), 99-111.

[R] M. Reid, *Canonical 3-folds*, Journées de Géométrie Algébrique d'Angers, A. Beauville (editor), Sijthoff and Noordhoff, Alphen aan den Rijn, 1980, pp. 273-310.

[Rr] I. Reider, *Vector bundles of rank 2 and linear systems on algebraic surfaces*, Ann. of Math. **127**(1988), 309-316.

[S] E. Stagnaro, *Canonical and pluricanonical adjoints to an algebraic hypersurface*, Doc. Math. J. DMV(ICM 1998), pp.60.

[T] S. G. Tankeev, *On n-dimensional canonically polarized varieties and varieties of fundamental type*, Izv. A. N. SSSR, Sér. Math. **35**(1971), 31-44.

[V] E. Viehweg, *Vanishing theorems*, J. reine angew. Math. **335**(1982), 1-8.

[X1] G. Xiao, *Finitude de l'application bicanonique des surfaces de type général*, Bull. Soc. Math. France **113**(1985), 23-51.

[X2] ——, *Linear bound for abelian automorphisms of varieties of general type*, J. reine angew. Math. **476**(1996), 201-207.